I. J. Castillioneus $D^{no.}$ *De* Montagny *V. C.* *Philosophiæ Professori in Academia* Lauzannensi, REGIÆ SOCIETATIS Londinensis *Membro dignissimo*, S^{ti} *Evangelii Ministro*, &c. &c. S. P. D.

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NEMO ignorat Newtonianam formulam, qua *Polynomium* quodcumque, ope binomii assumpti, ad quamvis potestatem extollitur; sed nemo, quod sciam, eam demonstravit. Hoc ego facere conatus meditatiunculas meas tibi æquissimo & optimo Judici mitto. *Tu, corrige, sodes, hoc dic, hocque, parum claris lucem dare coge, arguito ambigue dictum, mutanda notato.*

Continet hoc Problema tria prorsus diversa, quæ cum diversimode gignantur, & cum optima demonstratio e rei natura, vel genesi ducatur, diversa quoque probatione sunt confirmanda: Siquidem index est aut *integer*, aut *fractus*, uterque demum vel *positivus*, vel *negativus*.

1. Index sit *integer*, & *positivus*, tunc binomium ad potestatem cujus index est m elevare, nihil aliud est, quam toties binomium datum scribere, quoties unitas est in m, & omnia hæc binomia invicem ducere.

2. Si index est *fractus*, & *positivus*, binomium elevare ad potestatem $\frac{r}{n}$ est, datum binomium elevare ad potestatem r, &, hac potestate data, quærere quantitatem, quæ data ad potestatem n æquat ipsam dati binomii potestatem r.

N 3. Cum

3. Cum vero Index eſt *negativus*, ſive is *integer*, ſive *fractus*, ut binomium elevetur, facienda ſunt, quæ ſupra N°. 1. vel 2, & deinde per inventam poteſtatem unitas eſt dividenda.

Sumo Binomium $p+q$, ut indicet mihi quodvis Polynomium.

Inter p^m, & q^m tot ſunt medii Geometrici, in ratione $p.q$ quot unitates in $m-1$.

Hos terminos inventurus noto, quod p^m eſt ad q^m in ratione compoſita ipſius $p^m.1$, & $1.q^m$, ut & p ad q habet rationem compoſitam ex $p.1$, & ex $1.q$; ſed ſi fiant duæ ſeries poteſtatum, in quarum altera indices ipſius p decreſcant eadem proportione arithmetica, cujus differentia eſt 1, qua creſcunt in ſecunda ſerie indices ipſius q, habebitur ſeries continue proportionalium in ratione $p.1$, & $1.q$.

$$\text{Sic } p.1 :: p^m.p^{m-1}.p^{m-2}.p^{m-3}.p^{m-4}.\ldots.p^{m-m}=p^0=1$$
$$1.q :: 1.q.\quad q^2.\ q^3.\ q^4.\ \ldots\ldots\ q^m.$$

Ergo terminis reſpondentibus invicem ductis

$$p.q :: p^m.p^{m-1}q.p^{m-2}q^2.p^{m-3}q^3.p^{m-4}q^4\ldots q^m$$

Nunc dico $\overline{p+q}|^m$ componi ex terminis ſupra inventis, ut facile ex geneſi probatur.

Ergo omnes termini, qui ſunt in $\overline{p+q}|^m$ ordine diſpoſiti ſunt in proportione continua.

Et quidem duo quivis ſeſe immediate ſequentes ſunt, ut primus binomialis radicis terminus ad ſecundum.

Quod patet ex geneſi, nam p aliquoties ductum eſt ad q toties ductum in p, ut $p.q$.

Igitur omnium numerus eſt $m+1$; ſed & in ſerie arithmetica decreſcente $m.m-1.m-2.\ldots\ldots 0-$ termini ſunt numero $m+1$, aut creſcente $0.1.2.3.\ldots\ldots m$;

. . . . . m; ergo termini componentes $\overline{p+q}|^m$ debent habere indices hos, aut esse $p^m. p^{m-1}q.$ . . . . . q^m.

Atqui ex legibus multiplicationis numerus terminorum debet esse $2^m > m+1$, ergo in hoc facto aliqui termini repetiti debent inveniri.

Vulgaria facta (ea, nempe, quorum multiplicans & multiplicandum constat quantitatibus diversis) omnes continent diversos terminos, quia omnes formantur diversis factoribus. In potestatibus ergo dispiciendum quinam termini diversi essent, nisi factores semper essent iidem, & quot ex diversis restitutione literarum æquales fiant; sic enim reperiemus quoties quisque in potestate repeti debeat.

Jam patet, quod si factores semper essent diversi, diversi quoque essent omnes termini in producto.

Quod cum primus in producto non fiat nisi ex primis multiplicantium, & ultimus illius ex horum ultimis, semper hæc facta erunt diversa, quamvis binomia facientia sint eadem, quia primus binomii terminus differt a secundo.

Quod ex cæteris aliqui possunt fieri æquales, quia conflantur ex primis facientium ductis in secundos, & diversimode junctis.

Igitur quærendum est, quot diversis modis jungi possint quantitates, quarum numerus datus est.

In casu nostro index rerum est m, res diversæ duæ, quarum una repetitur vicibus s, altera t, ita ut $s+t = m$; ergo numerus permutationum erit

$$\frac{m.m-1.m-2.m-3 \;.\;.\;.\;.\;.\;.\;.\; 1}{s.s-1.s-2 \;.\;.\;.\; 1.t.t-1.t-2.t-3 \;.\;.\;.\; 1}$$

Sic sit $t=1$, $s=m-1$, terminus erit $p^{m-1}q$, & ejus coefficiens $\dfrac{m.m-1.m-2.m-3 \;.\;.\;.\;.\; 1}{m-1.m-2.m-3 \;.\;.\;.\;.\;.\; 1} = m$.

 Sit

Sit $t=3$, $s=m-3$; habebitur coefficiens ipſius $p^{m-3}q^3$, $= \frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}.\overline{m-4}\ldots\ldots 1}{1.2.3.\overline{m-3}.\overline{m-4}.\overline{m-5}\ldots\ldots 1} = \frac{m.\overline{m-1}.\overline{m-2}}{1.2.3.}$, & ſic de cæteris.

Si quis forte dubitet, an ſuperior demonſtratio evincat omnes terminos neceſſario formari tot modis, quibus poſſunt, & contendat eam tantum oſtendere id accidere poſſe, hoc reſponſi ferat.

Certe $\overline{p+q}|^m = \overline{p+q} \times \overline{p+q}|^{m-1}$; ſed inter hujus terminos ſunt $p^{m-n-1}q^n$, & $p^{m-n}q^{n-1}$, quæ neceſſario ducentur in p & q, & $p^{m-n-1}q^n \times p = p^{m-n}q^n = p^{m-n}q^{n-1} \times q$, ergo $p^{m-n}q^n$ omnibus modis poſſibilibus factum erit in $\overline{p+q}|^m$, ſi $p^{m-n-1}q^n$ & $p^{m-n}q^{n-1}$ ſint genita quot modis poſſunt in $\overline{p+q}|^{m-1}$; quod neceſſario erit, ſi $p^{m-n-2}q^n$, & $p^{m-n}q^{n-2}$ ſint in inferiori poteſtate $\overline{p+q}|^{m-2}$, & ſic ſemper uſque ad quadratum in quo pp, pq, & qq habentur, efficta tot quot poſſunt modis (4. II. *Euclid.*) ergo & in ſuperioribus.

Hoc ratiocinium monet, ut idem etiam ſic oſtendam, ratione paulo diverſa.

Jam primi coefficientem eſſe unitatem demonſtravimus.

Secundus terminus $p^{m-1}q$ conficitur ex $p^{m-2}q \times p$, & $p^{m-1} \times q$, id eſt, ex primo radicis in ſecundum ipſius $\overline{p+q}|^{m-1}$, & ex ſecundo radicis in primum $\overline{p+q}|^{m-1}$, ergo in $\overline{p+q}|^m$ adeſt $p^{m-1}q$ ſemel, plus toties, quoties ſecundus eſt in $\overline{p+q}|^{m-1}$, qui ibi eſt ſemel, plus toties, quoties ſecundus in $\overline{p+q}|^{m-2}$, qui rurſus ibi eſt ſemel plus

plus toties, quoties secundus est in $\overline{p+q}|^{m-2}$, & sic semper donec deveniatur ad $\overline{p+q}|^{m-m}$, ubi semel est secundus; ergo quærenda est summa tot unitatum, quot sunt in m, quæ est m.

Item tertius $p^{m-2}qq$ conficitur ex $p^{m-3}qq \times p$, tertio $\overline{p+q}|^{m-1}$ in primum radicis, & ex $p^{m-2}q \times q$ secundo ipsius $\overline{p+q}|^{m-1}$ in secundum radicis; ergo $\overline{p+q}|^{m}$ continebit $p^{m-2}qq$ quoties secundus continetur in $\overline{p+q}|^{m-1}$, id est, $m-1$ vices, plus toties quoties ibidem astat tertius, id est, quoties secundus est in $\overline{p+q}|^{m-2}$ $(m-2)$ plus quoties ibi est tertius, qui rursus est quoties secundus est in $\overline{p+q}|^{m-3}$ $(m-3)$ plus quoties ibi est tertius, atque ita porro donec perveniamus ad $\overline{p+q}|^{2}$ ubi semel est tertius, aut ad $p+q$, ubi tertius nullus est; nam semper quærenda est summa progressionis arithmeticæ $m-1 . m-2 . m-3 . \ldots\ldots 1$, aut $m-1 . m-2 . \ldots\ldots\ldots 0$, in illa numerus terminorum est $m-1$, in hac m, ut patet; quare hæc summa $= \overline{m-1+1} \times \frac{m-1}{2} = m \times \frac{m-1}{2} = \overline{m-1+0} \times \frac{m}{2}$.

Eodem pacto coefficientes reliquorum terminorum probabuntur efficere seriem in qua secundæ differentiæ sunt in progressione arithmetica, *&c.*

Unde semper, ubi m est integer, & positivus, formula erit $p^m + mp^{m-1}q + \frac{m . \overline{m-1}}{2} p^{m-2}qq +$

$$\frac{m . \overline{m-1} . \overline{m-2}}{2 . 3} p^{m-3}q^3 + \frac{m . \overline{m-1} . \overline{m-2} . \overline{m-3}}{2 . 3 . 4} p^{m-4}q^4$$

$$+ \frac{m . \overline{m-1} . \overline{m-2} . \overline{m-3} . \overline{m-4}}{2 . 3 . 4 . 5 .} p^{m-5}q^5, \text{ \&c.}$$

Si

Si fiat $p+q=p\times\overline{1+\frac{q}{p}}$, hinc orietur ipsissima Newtoni formula; nam $\overline{p+q}|^{m}=p^{m}\times\overline{1+\frac{q}{p}}|^{m}=$

$$p^{m}\times\overline{1+\frac{m}{1}\times\frac{p}{q}+\frac{m.\overline{m-1}}{1.2}\times\frac{p^{2}}{q^{2}}+\frac{m.\overline{m-1}.\overline{m-2}}{1.2.3.}\times\frac{p^{3}}{q^{3}},\&c.}=$$

(si A, B, C, D, &c. ponantur æquare primum, secundum, tertium, quartum, &c. cum suis quemque coefficientibus) $p^{m}\times\overline{1+mA\frac{q}{p}+\frac{m-1}{2}B\frac{q}{p}+\frac{m-2}{3}C\frac{q}{p}}$

$$\overline{+\frac{m-3}{4}D\frac{q}{p}+\frac{m-4}{5}E\frac{q}{p}+\frac{m-5}{6}F\frac{q}{p},\&c.}$$

Quæramus nunc formulam elevandi ejusdem binomii ad potestatem $\frac{r}{n}$, ubi r & n sunt numeri integri, & ambo vel positivi, vel negativi.

Jam $p.q::p^{\frac{r}{n}}.x=\frac{p^{\frac{r}{n}}q}{p}=p^{\frac{r}{n}-1}q$, quare termini erunt $p^{\frac{r}{n}}.p^{\frac{r}{n}-1}q.p^{\frac{r}{n}-2}qq.p^{\frac{r}{n}-3}q^{3}$, &c.

Coefficientes inveniendi sint A, B, C, D, E, ita ut tota $\overline{p+q}|^{\frac{r}{n}}$ radix $=Ap^{\frac{r}{n}}+Bp^{\frac{r}{n}-1}q+Cp^{\frac{r}{n}-2}qq+Dp^{\frac{r}{n}-3}q^{3}+Ep^{\frac{r}{n}-4}q^{4}$, &c. ergo $\overline{p+q}|^{r}(p^{r}+rp^{r-1}q+\frac{r.\overline{r-1}}{2}p^{r-2}qq+\frac{r.\overline{r-1}.\overline{r-2}}{2.3}p^{r-3}q^{3}$, &c.) $=\overline{Ap^{\frac{r}{n}}+Bp^{\frac{r}{n}-1}q+Cp^{\frac{r}{n}-2}qq,\&c.}|^{n}=A^{n}p^{r}+nA^{n-1}Bp^{r-1}q+nA^{n-1}Cp^{r-2}qq+nA^{n-1}Dp^{r-3}q^{3}+nA^{n-1}Ep^{r-4}q^{4}$ &c.

$+n.n-$

$+\frac{n.\overline{n-1}}{2}A^{n-2}B^2p^{r-2}qq+n.\overline{n-1}\,A^{n-2}BCp^{r-3}q^3+$
$n.\overline{n-1}\,A^{n-2}BDp^{r-4}q^4$ &c. $+\frac{n.\overline{n-1}.\overline{n-2}}{2.3}A^{n-3}B^3p^{r-3}q^3$
$+\frac{n.\overline{n-1}.\overline{n-2}}{2}A^{n-3}B^2Cp^{r-4}q^4$ &c. $+$
$\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3.4}A^{n-4}B^4p^{r-4}q^4$. Atque ideo collatis terminis $1=A^n=A^{n-1}=A^{n-2}$ &c. $nB=r$, & $B=\frac{r}{n}$, $nC+\frac{n.\overline{n-2}}{2}\times\frac{rr}{nn}=\frac{r.\overline{r-1}}{2}$, & $C=\frac{r.\overline{r-n}}{2.nn}$, $nD+$
$n.\overline{n-1}\times\frac{r}{n}\times\frac{r.\overline{r-n}}{2nn}+\frac{n.\overline{n-1}.\overline{n-2}}{2.3}\times\frac{r^3}{n^3}=\frac{r.\overline{r-1}.\overline{r-2}}{2.3}$, &
$D=\frac{r.\overline{r-n}.\overline{r-2n}}{2.3.n^3}$ &c.

Si ergo faciamus $\frac{r}{n}=m$, & primum terminum A, &c. revivet prior formula, & $\overline{p+q}|^{\frac{r}{n}}=\overline{p+q}|^{m}=p^m\times$
$1+mA\frac{q}{p}+\frac{m-1}{2}B\frac{p}{q}+\frac{m-2}{3}C\frac{q}{p}$ &c.

Extollendum sit binomium $p+q$ ad negativam potestatem, seu perfectam, seu imperfectam $-s$.

Jam $\overline{p+q}|^{-s}=\frac{1}{\overline{p+q}|^{s}}=\frac{1}{p^s+sp^{s-1}q+\frac{s.\overline{s-1}}{2}p^{s-2}qq\text{ &c.}}$

$=$(per divisionem) $\frac{1}{p}-\frac{s\,p^{s-1}q}{p^{2s}}-\frac{s.\overline{s-1}}{2}\times\frac{p^{s-2}qq}{p^{2s}}$

$-s.s-$

$$-\frac{s.s-1.s-2}{2.3}\times\frac{p^{s-3}q^3}{p^{2s}}-\frac{s.s-1.s-2.s-3}{2.3.4}\times\frac{p^{s-4}q^4}{p^{2s}}=$$

$$p^{-s}-sp^{-s-1}q-\frac{s.s-1}{2}p^{-s-2}qq.$$

Ex hac formula facile, superiorum vestigiis insistendo, eruitur solemnis & generalissima $p^m \times 1 + mA\frac{q}{p}+\frac{m-1}{2}B\frac{q}{p}$ &c.

Non injucundum puto, quod in hac formula, si $m=-2$, coefficientes erunt numeri naturales, si $m=-3$, trigonales, pyramidales, si $m=-4$ &c.

Cæterum constat hanc formulam semper dare seriem infinitam; siquidem (si m exponit numerum positivum) ultimus terminus esse deberet q^{-m}; sed $p.q::p^{-m}.p^{-m-1}::p^{-m-1}q.p^{-m-2}qq$, &c. ergo ratio ipsius $p^{-m}.q^{-m}$ componi deberet ex aliquibus rationibus $p.q$, quod fieri nequit, quia $p^{-m}.q^{-m}::\frac{1}{p^m}.\frac{1}{q^m}::q^m.p^m$ in ratione composita ex reciprocis ipsius $p.q$.

Quod & aliter demonstratur, indices ipsius p faciunt progressionem arithmeticam, cujus termini $-m$, $-m-1$, $-m-2$, &c. negativi quidem sunt, sed crescunt, aut ab 3 recedunt; atqui ultimus terminus debet esse $q^{-m}=p^0q^{-m}$, ergo nunquam ad illud devenietur.

Viviaci,
postridie Id. *Septemb.*
CIↃ IↃCCXXXXI.

II. *Two*